

Steiner triple systems without parallel classes

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Abstract

We construct Steiner triple systems without parallel classes for an infinite number of orders congruent to 3 (mod 6). The only previously known examples have order 15 or 21.

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A *Steiner triple system* of order v is a pair (V, \mathcal{B}) where V is a v -set of *points* and \mathcal{B} is a collection of 3-subsets of V , called *triples*, such that each (unordered) pair of points occurs in exactly one triple. Steiner triple systems are fundamental in design theory. Kirkman [4] proved in 1847 that there exists a Steiner triple system of order v if and only if $v \equiv 1$ or $3 \pmod{6}$ (see [3]).

A *parallel class* in a Steiner triple system (V, \mathcal{B}) is a subset of \mathcal{B} that partitions V . Of course, only Steiner triple systems of order congruent to 3 (mod 6) may contain parallel classes. It is a long-standing conjecture, dating back to at least 1984, that systems without parallel classes exist for all orders congruent to 3 (mod 6) except 3 and 9 (see [3]). In spite of this, the only previously known examples without parallel classes have order 15 or 21. The unique Steiner triple system of order 9 has a parallel class, 10 of the 80 nonisomorphic Steiner triple systems of order 15 have no parallel classes (see [3]), and 12 of the 1772 nonisomorphic 4-rotational Steiner triple systems of order 21 have no parallel classes [5].

In this paper we construct Steiner triple systems without parallel classes for an infinite number of orders congruent to 3 (mod 6). Our construction is similar to one used by Schreiber [10] and Wilson [11]. Problems concerning parallel classes in Steiner triple systems are well studied. Ray-Chaudhuri and Wilson [8] famously proved that for every order congruent to 3 (mod 6) there exists a Steiner triple system whose block set can be partitioned into parallel classes. The study of parallel classes in Steiner triples also relates to problems concerning matchings in certain hypergraphs, and Alon et al. [1] give a lower bound on the number of disjoint triples in any Steiner triple system. Also see [7] and the surveys [3] and [9].

The existence of *almost parallel classes* (sets of disjoint triples containing all the points except one) in Steiner triple systems of order congruent to 1 (mod 6) has also been investigated. It is conjectured that for all $v \equiv 1 \pmod{6}$ except $v = 13$ there exists a Steiner triple system of order v with no almost parallel class [9]. Two sparse infinite families of such Steiner triple systems have been found, one by Wilson (see [9]) and one by the authors [2]. Also see [6] for a recent related result.

For a prime p , let $\text{ord}_p(x)$ denote the multiplicative order of x in \mathbb{Z}_p . Let \mathcal{P} be the set of odd primes given by $p \in \mathcal{P}$ if and only if $\text{ord}_p(-2) \equiv 0 \pmod{4}$. Observe that \mathcal{P} contains all of the infinitely many primes congruent to 5 (mod 8), no primes congruent to 3 or 7 (mod 8), and some primes congruent to 1 (mod 8). Clearly, if $p \equiv 3 \pmod{4}$ is prime, then $\text{ord}_p(-2) \not\equiv 0 \pmod{4}$. If $p \equiv 5 \pmod{8}$ is prime, then -2 is a quadratic non-residue in \mathbb{Z}_p by the law of quadratic reciprocity and it follows that $\text{ord}_p(-2) \equiv 0 \pmod{4}$. Our theorem states that there is a Steiner triple system of order v with no parallel class whenever $v \equiv 27 \pmod{30}$ and $p \in \mathcal{P}$ for each prime divisor p of $v - 2$. Thus, for any list p_1, \dots, p_t of (not necessarily distinct) primes from \mathcal{P} in which the number of primes congruent to 5 (mod 6) is odd, we obtain a Steiner triple system of order $v = 5p_1 \cdots p_t + 2$ with no parallel class.

Theorem 1. *Let \mathcal{V} be the set of positive integers given by $v \in \mathcal{V}$ if and only if $v \equiv 27 \pmod{30}$ and $\text{ord}_p(-2) \equiv 0 \pmod{4}$ for each prime divisor p of $v - 2$. Then \mathcal{V} is infinite and for each $v \in \mathcal{V}$ there exists a Steiner triple system of order v having no parallel class.*

Proof. It is clear from the remarks preceding the theorem that \mathcal{V} is infinite. We now show that if $v \in \mathcal{V}$, then there exists a Steiner triple system of order v having no parallel class. Let $v \in \mathcal{V}$, define n by $v = 5n + 2$ (note that n is an odd integer because $v \equiv 27 \pmod{30}$), and let $n = p_1 p_2 \cdots p_t$ be the prime factorisation of n . By our hypotheses $\text{ord}_{p_i}(-2) \equiv 0 \pmod{4}$ for $i = 1, 2, \dots, t$. Let $p_0 = 5$ and let $H = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_t}$ and $G = \mathbb{Z}_{p_0} \times H$ be additive groups with identities $0_H = (0, 0, \dots, 0)$ and $0_G = (0, 0_H)$. If $m = \frac{a}{b}$ where a and b are integers such that $\gcd(b, p_i) = 1$ for $i = 0, \dots, t$, and $g = (g_0, g_1, g_2, \dots, g_t) \in G$, then we define mg by $mg = (ab^{-1}g_0, ab^{-1}g_1, \dots, ab^{-1}g_t)$ where $ab^{-1}g_i$ is evaluated in the field \mathbb{Z}_{p_i} for $i = 0, \dots, t$.

Let

$$\mathcal{B}^0 = \{\{g, g', g''\} : g, g', g'' \in G, |\{g, g', g''\}| = 3, g + g' + g'' = 0\}.$$

Observe that each pair of distinct elements of G occurs in at most one triple of \mathcal{B}^0 , and that the pairs that occur in no triple of \mathcal{B}^0 are exactly the edges of the graph X with vertex set $V(X) = G$ and edge set

$$E(X) = \{\{g, -2g\} : g \in G \setminus \{0_G\}\}.$$

Thus, 0_G is an isolated vertex in X and each vertex $g \in G \setminus \{0_G\}$ has exactly 2 neighbours in X , namely $-\frac{1}{2}g$ and $-2g$. That is, X is a collection of disjoint cycles together with an isolated vertex. To each edge of X assign a weight equal to the sum in G of its endpoints. It follows that for each $g \in G \setminus \{0_G\}$ there is a unique edge in X having weight g (namely the edge $\{-g, 2g\}$). Moreover, the edge of X having weight g has an adjacent edge of weight $-2g$. Henceforth we denote the edges of X by their weights and describe the cycles of X by listing their edges in cyclically ordered tuples. So the cycle containing an arbitrary edge g of X has vertices $-g, 2g, \dots, \frac{1}{2}g$, edges $\{-g, 2g\}, \{2g, -4g\}, \dots, \{\frac{1}{2}g, -g\}$, and is denoted by

$$(g, -2g, \dots, -\frac{1}{2}g).$$

Let $\theta : H \setminus \{0_H\} \rightarrow H \setminus \{0_H\}$ be given by $\theta(h) = -2h$, and let \mathcal{Y} be the set of orbits of θ . Since $\text{ord}_{p_i}(-2) \equiv 0 \pmod{4}$ for $i = 1, 2, \dots, t$, we have $|Y| \equiv 0 \pmod{4}$ for each $Y \in \mathcal{Y}$. Thus, each cycle of X except

$$C = ((1, 0_H), (3, 0_H), (4, 0_H), (2, 0_H))$$

is of the form

$$((a, h), (3a, \theta(h)), (4a, \theta^2(h)), \dots, (2a, \theta^{|Y|-1}(h)))$$

for some $a \in \mathbb{Z}_5$, $h \in Y$ and $Y \in \mathcal{Y}$. Note that $E(X) \setminus E(C) = \mathbb{Z}_5 \times (H \setminus \{0_H\})$.

We claim that there is a function $\gamma : \mathbb{Z}_5 \times (H \setminus \{0_H\}) \rightarrow \{1, 2\}$ such that for all $g, g' \in \mathbb{Z}_5 \times (H \setminus \{0_H\})$

- (1) $\gamma(g) \neq \gamma(-2g)$; and
- (2) if $g + g' \in \mathbb{Z}_5 \times \{0_H\}$, then $\gamma(g) = \gamma(g')$.

For each $Y \in \mathcal{Y}$, let $-Y = \{-h : h \in Y\}$. It is clear that if $Y \in \mathcal{Y}$, then $-Y \in \mathcal{Y}$. Thus, either $-Y = Y$ or Y and $-Y$ are distinct orbits in \mathcal{Y} . For each $Y \in \mathcal{Y}$, choose a representative element h_Y ensuring that if Y and $-Y$ are distinct, then $h_{-Y} = -h_Y$. Write $H \setminus \{0_H\}$ as a union of disjoint sets H_1 and H_2 given by

$$H_1 = \bigcup_{Y \in \mathcal{Y}} \{\theta^i(h_Y) : i = 0, 2, \dots, |Y| - 2\} \quad \text{and} \quad H_2 = \bigcup_{Y \in \mathcal{Y}} \{\theta^i(h_Y) : i = 1, 3, \dots, |Y| - 1\}.$$

Since $|Y|$ is even for each $Y \in \mathcal{Y}$, H_1 and H_2 are well defined. For each $a \in \mathbb{Z}_5$ and each $h \in H \setminus \{0_H\}$ define the function $\gamma : \mathbb{Z}_5 \times (H \setminus \{0_H\}) \rightarrow \{1, 2\}$ by $\gamma((a, h)) = 1$ if $h \in H_1$ and $\gamma((a, h)) = 2$ if $h \in H_2$. It is clear that γ satisfies (1). It also satisfies (2) because $|Y| \equiv 0 \pmod{4}$ for each $Y \in \mathcal{Y}$, and because we have $h_{-Y} = -h_Y$ when Y and $-Y$ are distinct.

We now construct a Steiner triple system of order $v = |G| + 2$ which we will prove has no parallel class. Let ∞_1 and ∞_2 be two new points (not in G) and let

$$\mathcal{B}^\infty = \{\{-g, 2g, \infty_{\gamma(g)}\} : g \in \mathbb{Z}_5 \times (H \setminus \{0_H\})\}.$$

Also, let $(\{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), \infty_1, \infty_2\}, \mathcal{B}^*)$ be any Steiner triple system of order 7 such that $\{(0, 0), \infty_1, \infty_2\} \notin \mathcal{B}^*$. It is easily seen that

$$\mathcal{S} = (G \cup \{\infty_1, \infty_2\}, \mathcal{B}^0 \cup \mathcal{B}^\infty \cup \mathcal{B}^*)$$

is a Steiner triple system of order v . We now show that \mathcal{S} has no parallel class.

For a contradiction, suppose \mathcal{C} is a parallel class of \mathcal{S} . Define the weight of each triple of \mathcal{S} to be the sum in G of its points, where the points ∞_1 and ∞_2 are treated as 0_G . Since the sum of the elements of G is 0_G , the sum of the weights of the triples in \mathcal{C} is also 0_G . Now note the following properties concerning weights of triples which we will use in the arguments that follow. Every triple in \mathcal{B}^0 has weight 0_G , every triple in \mathcal{B}^* has weight in $\mathbb{Z}_5 \times \{0_H\}$, and every triple in \mathcal{B}^∞ has weight in $\mathbb{Z}_5 \times (H \setminus \{0_H\})$. The weight of the triple that contains both ∞_1 and ∞_2 is not 0_G .

If the triple that contains both ∞_1 and ∞_2 is in \mathcal{C} , then no other triple of \mathcal{B}^* is in \mathcal{C} (because any two triples of a Steiner triple system of order 7 intersect) and no triple of \mathcal{B}^∞ is in \mathcal{C} (because every triple of \mathcal{B}^∞ contains either ∞_1 or ∞_2). This means that the remaining triples of \mathcal{C} are from \mathcal{B}^0 . But this is a contradiction because every triple in \mathcal{B}^0 has weight 0_G , and the weight of the triple that contains both ∞_1 and ∞_2 is not 0_G . We conclude that \mathcal{C} contains distinct triples T_1 and T_2 such that $\infty_1 \in T_1$ and $\infty_2 \in T_2$, and that the remaining triples of \mathcal{C} are all from $\mathcal{B}^0 \cup \mathcal{B}^*$.

Since every triple in $\mathcal{B}^0 \cup \mathcal{B}^*$ has weight in $\mathbb{Z}_5 \times \{0_H\}$, the sum of the weights of T_1 and T_2 is also in $\mathbb{Z}_5 \times \{0_H\}$. This means that either both T_1 and T_2 are in \mathcal{B}^* or both are in \mathcal{B}^∞ (because every triple in \mathcal{B}^* has weight in $\mathbb{Z}_5 \times \{0_H\}$, and every triple in \mathcal{B}^∞ has weight in $\mathbb{Z}_5 \times (H \setminus \{0_H\})$). The triples T_1 and T_2 cannot both be in \mathcal{B}^* because any two triples of \mathcal{B}^* intersect. On the other hand, if T_1 and T_2 are both in \mathcal{B}^∞ , then the fact that the sum of the weights of T_1 and T_2 is in $\mathbb{Z}_5 \times \{0_H\}$ contradicts Property (2) of γ . We conclude that \mathcal{S} has no parallel class. \square

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